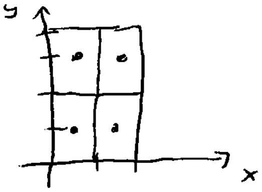
Midterm 2 Review Solutions

L1

① a)

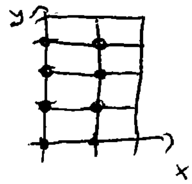


Points	$f(x,y) = x+2y^2$
$(\frac{1}{2},1)$	$5/2$
$(\frac{3}{2},1)$	$7/2$
$(\frac{1}{2},3)$	$37/2$
$(\frac{3}{2},3)$	$39/2$

$\Delta A = \Delta x \Delta y = 1 \cdot 2 = 2$

$$V \approx 2\left(\frac{5}{2}+\frac{7}{2}+\frac{37}{2}+\frac{39}{2}\right) = 88$$

b)



Points	$f(x,y) = x+2y^2$
(0,0)	0
(1,0)	1
(0,1)	2
(1,1)	3
(0,2)	8
(1,2)	9
(0,3)	18
(1,3)	19

$\Delta A = \Delta x \Delta y = 1 \cdot 1 = 1$

$V \approx 1(0+1+2+3+8+9+18+19)$

$= 60$

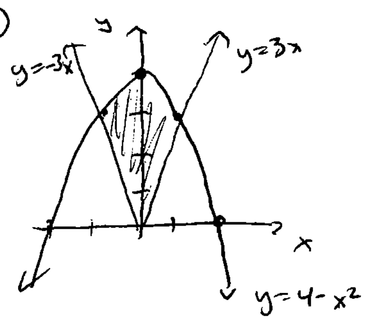
c) $\int_0^2 \int_0^4 x+2y^2 \, dy\, dx$

$\int_0^2 \left(xy + \frac{2}{3}y^3 \Big|_0^4\right) dx$

$\int_0^2 4x + \frac{128}{3} \, dx$

$2x^2 + \frac{128}{3}x \Big|_0^2 = 8 + \frac{256}{3} = \frac{280}{3}$

②



$y = |3x| = \begin{cases} 3x & x \geq 0 \\ -3x & x < 0 \end{cases}$

$$\int_0^1 \int_{3x}^{4-x^2} f(x,y)\, dy\, dx + \int_{-1}^0 \int_{-3x}^{4-x^2} f(x,y)\, dy\, dx$$

or

$x^2 = 4-y$

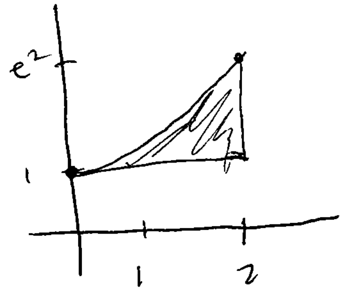
$x = \pm\sqrt{4-y}$

$$\int_3^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\, dx\, dy + \int_0^3 \int_{-\frac{y}{3}}^{y/3} f(x,y)\, dx\, dy$$

③ a) $\int_1^{e^2} \int_{\ln(y)}^{2} \frac{e^{x^2}}{y}\, dx\, dy$

Cannot integrate in this order so need to change order, first draw region.

$x = \ln(y)$
$\downarrow$
$e^x = y$



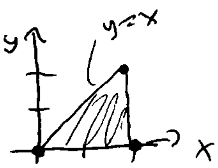
$= \int_0^2 \int_1^{e^x} \frac{e^{x^2}}{y}\, dy\, dx = \int_0^2 \left(\ln(y)\, e^{x^2} \Big|_1^{e^x} \right) dx$

$= \int_0^2 \ln(e^x) e^{x^2} - \ln(1) e^{x^2}\, dx = \int_0^2 x e^{x^2}\, dx = \int_0^4 \frac{1}{2} e^u\, du$

$u = x^2$
$du = 2x\, dx$

$= \frac{1}{2} e^u \Big|_0^4 = \frac{1}{2} e^4 - \frac{1}{2} e^0 = \frac{1}{2} e^4 - \frac{1}{2}$

b)



Need to int. with respect to y first.

$\int_0^2 \int_0^x y \sin(x^3)\, dy\, dx = \int_0^2 \left(\frac{1}{2} y^2 \sin(x^3) \Big|_0^x \right) dx$

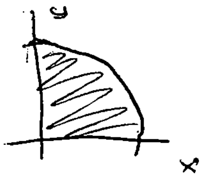
$= \int_0^2 \frac{1}{2} x^2 \sin(x^3)\, dx = \frac{1}{6} \int_0^8 \sin(u)\, du = -\frac{1}{6} \cos(u) \Big|_0^8 =$

$u = x^3$
$du = 3x^2\, dx$

$-\frac{1}{6} \cos(8) + \frac{1}{6}$

c) $\int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx = \int_0^1 \left(\frac{1}{2} xy^2 \Big|_0^{\sqrt{x}} \right) dx = \int_0^1 \frac{1}{2} x^2\,dx = \frac{1}{6} x^3 \Big|_0^1 = \frac{1}{6}$

d)



This can be done in either rectangular or polar coordinates.

In rectangular, easier to integrate y first

Rectangular

$$\int_0^1 \int_0^{\sqrt{1-x^2}} x^{1/3} y\,dy\,dx$$

$$\int_0^1 \frac{1}{2} x^{1/3} y^2 \Big|_0^{\sqrt{1-x^2}} dx$$

$$\int_0^1 \frac{1}{2} x^{1/3} (1-x^2)\,dx$$

$$\int_0^1 \frac{1}{2} x^{1/3} - \frac{1}{2} x^{7/3}\,dx$$

$$\frac{3}{8} x^{4/3} - \frac{3}{20} x^{10/3} \Big|_0^1$$

$$\frac{3}{8} - \frac{3}{20}$$

$$\frac{9}{40}$$

Polar

$$\int_0^{\pi/2} \int_0^1 r^{1/3} \cos^{1/3}\theta \, r \sin\theta \, r\,dr\,d\theta$$

$$\int_0^{\pi/2} \int_0^1 r^{7/3} \cos^{1/3}\theta \sin\theta\,dr\,d\theta$$

$$\int_0^{\pi/2} \left(\frac{3}{10} r^{10/3} \cos^{1/3}\theta \sin\theta \Big|_0^1 \right) d\theta$$

$$\int_0^{\pi/2} \frac{3}{10} \cos^{1/3}\theta \sin\theta\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$-\int_1^0 \frac{3}{10} u^{1/3}\,du$$

$$\int_0^1 \frac{3}{10} u^{1/3}\,du$$

$$\frac{9}{40} u^{4/3} \Big|_0^1 = \frac{9}{40}$$

e) $\int_0^1 \int_y^{\sqrt{2y-y^2}} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$ ← cannot integrate as is, x^2+y^2 terms suggest switch to polar

$x = \sqrt{2y-y^2}$

$x^2 = 2y - y^2$

$y^2 - 2y + x^2 = 0$

$y^2 - 2y + 1 + x^2 = 1$

$(y-1)^2 + x^2 = 1$

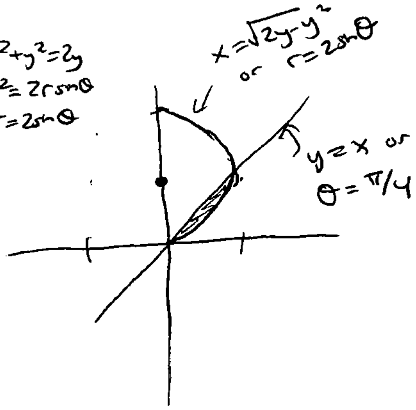
circ. rad 1

center $(0,1)$

$x^2+y^2 = 2y$

$r^2 = 2r\sin\theta$

$r = 2\sin\theta$



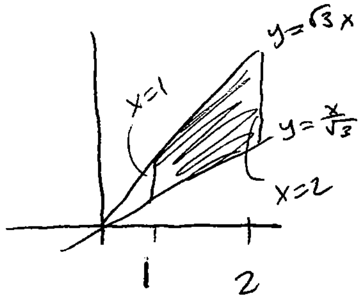
$$\int_0^{\pi/4} \int_0^{2\sin\theta} \frac{1}{\cancel{r}} \cancel{r}\,dr\,d\theta$$

$$\int_0^{\pi/4} \int_0^{2\sin\theta} dr\,d\theta$$

$$\int_0^{\pi/4} 2\sin\theta\,d\theta$$

$$-2\cos\theta \Big|_0^{\pi/4} = -\sqrt{2} + 2$$

f) Polar



$y=\sqrt{3}x$

$\cancel{r}\sin\theta=\sqrt{3}\cancel{r}\cos\theta$

$\frac{\sin\theta}{\cos\theta}=\sqrt{3}$

$\tan\theta=\sqrt{3}$

$\theta=\frac{\pi}{3}$

$y=\frac{x}{\sqrt{3}}$

$\frac{\sin\theta}{\cos\theta}=\frac{1}{\sqrt{3}}$

$\tan\theta=\frac{1}{\sqrt{3}}$

$\theta=\pi/6$

$x=1$

$r\cos\theta=1$

$r=\frac{1}{\cos\theta}$

$x=2$

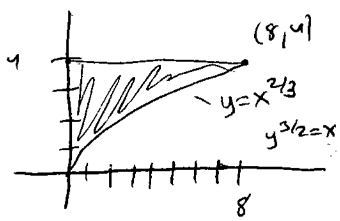
$r=\frac{2}{\cos\theta}$

$$\int_{\pi/6}^{\pi/3}\int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}}\frac{1}{r^3}\cdot r\,dr\,d\theta=\int_{\pi/6}^{\pi/3}\int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}}\frac{1}{r^2}\,dr\,d\theta$$

$$=\int_{\pi/6}^{\pi/3}\left(-\frac{1}{r}\Big|_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}}\right)d\theta=\int_{\pi/6}^{\pi/3}-\frac{1}{2}\cos\theta+\cos\theta\;d\theta=\int_{\pi/6}^{\pi/3}\frac{1}{2}\cos\theta\,d\theta$$

$$=\frac{1}{2}\sin\theta\Big|_{\pi/6}^{\pi/3}=\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=\frac{\sqrt{3}-1}{4}$$

g) Change order of integration



$$\int_0^4\int_0^{y^{3/2}}\frac{x}{\sqrt{x^2+y^3}}\,dx\,dy$$

$u=x^2+y^3$

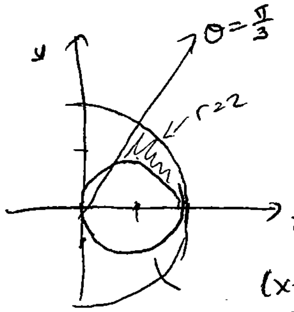
$du=2x\,dx$

$$\int_0^4\int_{x=0}^{x=y^{3/2}}\frac{1}{2}u^{-1/2}\,du\,dy=\int_0^4\left(u^{1/2}\Big|_{x=0}^{x=y^{3/2}}\right)dy$$

$$=\int_0^4\left(\sqrt{x^2+y^3}\Big|_{x=0}^{x=y^{3/2}}\right)dy=\int_0^4\sqrt{y^3+y^3}-\sqrt{y^3}\,dy=\int_0^4\sqrt{2}y^{3/2}-y^{3/2}\,dy$$

$$=(\sqrt{2}-1)\int_0^4 y^{3/2}\,dy=(\sqrt{2}-1)\cdot\frac{2}{5}y^{5/2}\Big|_0^4=(\sqrt{2}-1)\cdot 2\cdot\frac{32}{5}=\frac{64(\sqrt{2}-1)}{5}$$

h)



$(x-1)^2+y^2=1$

$x^2+y^2=2x\rightarrow r^2=2r\cos\theta$

$r=2\cos\theta$

$$\int_0^{\pi/3}\int_{2\cos\theta}^{2}r\sin\theta\cdot r\,dr\,d\theta=\int_0^{\pi/3}\left(\frac{1}{3}r^3\sin\theta\Big|_{2\cos\theta}^{2}\right)d\theta=\int_0^{\pi/3}\frac{8}{3}\sin\theta-\frac{8}{3}\cos^3\theta\sin\theta\,d\theta$$

$$=-\frac{8}{3}\cos\theta+\frac{2}{3}\cos^4\theta\Big|_0^{\pi/3}=\left(-\frac{8}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{16}\right)-\left(-\frac{8}{3}+\frac{2}{3}\right)$$

$$=\frac{17}{24}$$

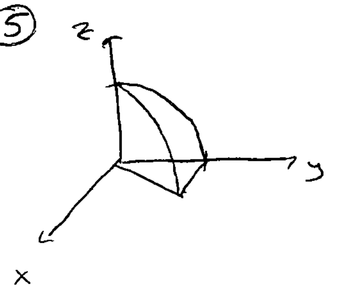
(4) The order of integration must be $dy\,dz\,dx$.

$$\int_0^1\int_0^x\int_x^{x+z} z\,dy\,dz\,dx = \int_0^1\int_0^x \left(zy\Big|_x^{x+z}\right) dz\,dx$$

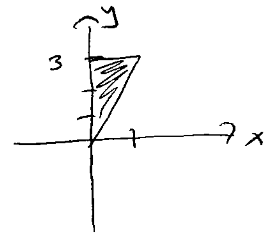
$$= \int_0^1\int_0^x zx+z^2-zx\,dz\,dx = \int_0^1\int_0^x z^2\,dz\,dx = \int_0^1 \left(\frac{1}{3} z^3\Big|_0^x\right)dx = \int_0^1 \frac{1}{3} x^3\,dx$$

$$= \frac{1}{12}x^4\Big|_0^1 = \frac{1}{12}$$

(5)



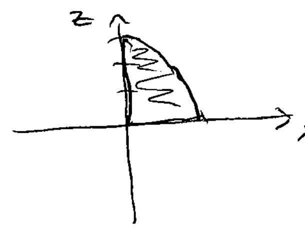
1) $$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx$$



2) $$\int_0^3\int_0^{y/3}\int_0^{\sqrt{9-y^2}} z\,dz\,dx\,dy$$

3) $$\int_0^3\int_0^{\sqrt{1-z^2/9}}\int_{3x}^{\sqrt{9-z^2}} z\,dy\,dx\,dz$$

4) $$\int_0^1\int_0^{\sqrt{9-9x^2}}\int_{3x}^{\sqrt{9-z^2}} z\,dy\,dz\,dx$$



$3x = \sqrt{9-z^2}$

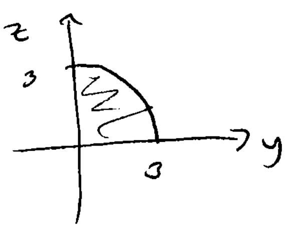
$9x^2 = 9 - z^2$

$9x^2 + z^2 = 9$

$x^2 + \frac{z^2}{9} = 1$

5) $$\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{y/3} z\,dx\,dy\,dz$$

6) $$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{y/3} z\,dx\,dz\,dy$$



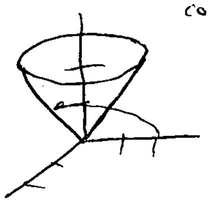
$y^2+z^2=9$

Integrate 1st one: $$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1\int_{3x}^3 \left(\frac{1}{2}z^2\Big|_0^{\sqrt{9-y^2}}\right) dy\,dx$$

$$= \int_0^1\int_{3x}^3 \frac{1}{2}(9-y^2)\,dy\,dx = \int_0^1 \frac{9}{2}y - \frac{1}{6}y^3\Big|_{3x}^3\,dx = \int_0^1 \left(\frac{27}{2}-\frac{9}{2}\right) - \left(\frac{27}{2}x - \frac{9}{2}x^3\right)dx$$

$$= \int_0^1 9 - \frac{27}{2}x + \frac{9}{2}x^3\,dx = 9x - \frac{27}{4}x^2 + \frac{9}{8}x^4\Big|_0^1 = 9 - \frac{27}{4} + \frac{9}{8} = \frac{72}{8} - \frac{54}{8} + \frac{9}{8} = \frac{27}{8}$$

(6)

cone  Switch to cylindrical coordinates:

$$\int_0^{2\pi}\int_0^2\int_r^2 r\cos\theta\, z\cdot r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2\int_r^2 r^2\cos\theta\, z\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 \tfrac{1}{2}r^2\cos\theta\, z^2\Big|_r^2\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 2r^2\cos\theta-\tfrac{1}{2}r^4\cos\theta\,dr\,d\theta=\int_0^{2\pi}\tfrac{2}{3}r^3\cos\theta-\tfrac{1}{10}r^5\cos\theta\Big|_0^2\,d\theta$$

$$=\int_0^{2\pi}\tfrac{16}{3}\cos\theta-\tfrac{16}{5}\cos\theta\,d\theta=\tfrac{32}{15}\int_0^{2\pi}\cos\theta\,d\theta=\tfrac{32}{15}\sin\theta\Big|_0^{2\pi}=0$$

(7) a) $\delta(x,y,z)=\sqrt{x^2+y^2+z^2}=\rho$

Use spherical coordinates, between $\rho=1$ and $\rho=\sqrt{2}$

$$\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt2}\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt2}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta$$

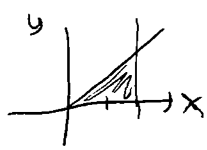
$$=\int_0^{2\pi}\int_0^{\pi}\tfrac{1}{4}\rho^4\sin\phi\Big|_1^{\sqrt2}\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}\sin\phi-\tfrac{1}{4}\sin\phi\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\tfrac{3}{4}\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}-\tfrac{3}{4}\cos\phi\Big|_0^{\pi}\,d\theta=\int_0^{2\pi}\tfrac{3}{4}+\tfrac{3}{4}\,d\theta$$

$$=\int_0^{2\pi}\tfrac{3}{2}\,d\theta=\tfrac{3}{2}\cdot2\pi=3\pi$$

b) Hard to draw, but $x=2$, $x=y$, $y=0$ gives triangular cylinder, cut off on bottom by $z=0$ and top by $z=2x$.

on xy-plane:

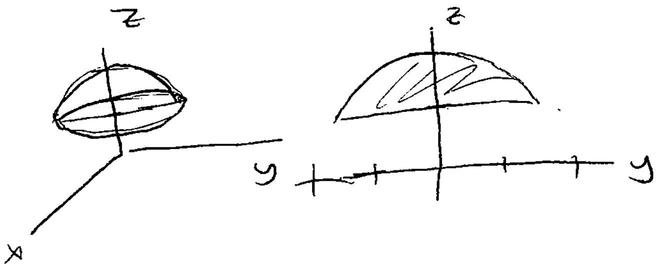


$$\int_0^2\int_0^x\int_0^{2x} y\,dz\,dy\,dx=\int_0^2\int_0^x yz\Big|_0^{2x}\,dy\,dx$$

$$=\int_0^2\int_0^x 2xy\,dy\,dx=\int_0^2 xy^2\Big|_0^x\,dx=\int_0^2 x^3\,dx$$

$$=\tfrac{1}{4}x^4\Big|_0^2=4$$

(8) Region is cap of sphere

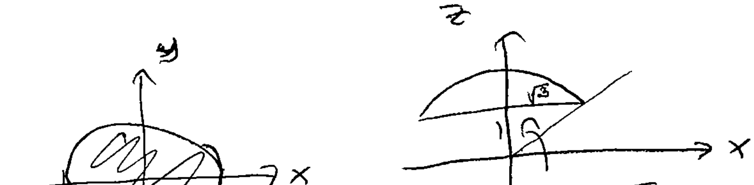


$x^2+y^2+z^2=4$ and $z=1$

$x^2+y^2+1=4$

$x^2+y^2=3$

bottom is circle radius $\sqrt{3}$ cent. at $(0,0)$



$\tan(\phi)=\sqrt{3}$

$\phi=\pi/3$

Integrate 1 to get volume.

$z=1$

$\rho\cos\phi=1$

$\rho=\frac{1}{\cos\phi}$

Cylindrical

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{4-r^2}-r\,dr\,d\theta$$

$$\int_0^{2\pi}\left(-\frac{1}{3}(4-r^2)^{3/2}-\frac{1}{2}r^2\Big|_0^{\sqrt{3}}\right)d\theta$$

$$\int_0^{2\pi}\left(-\frac{1}{3}-\frac{3}{2}\right)-\left(-\frac{8}{3}-0\right)d\theta$$

$$\int_0^{2\pi}\frac{5}{6}\,d\theta$$

$$\frac{5}{6}\cdot 2\pi$$

$$\frac{5\pi}{3}$$

Spherical

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{1}{\cos\phi}}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\rho^3\sin\phi\Big|_{\frac{1}{\cos\phi}}^{2}\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/3}\frac{8}{3}\sin\phi-\frac{\sin\phi}{3\cos^3\phi}\,d\phi\,d\theta$$

$$\int_0^{2\pi}-\frac{8}{3}\cos\phi-\frac{1}{6\cos^2\phi}\Big|_0^{\pi/3}\,d\theta$$

$$\int_0^{2\pi}\left(-\frac{8}{6}-\frac{4}{6}\right)-\left(-\frac{8}{3}-\frac{1}{6}\right)d\theta$$

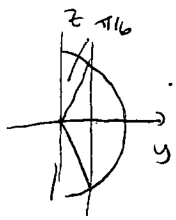
$$\int_0^{2\pi}\frac{5}{6}\,d\theta$$

$$\frac{5\pi}{3}$$

(9) **cylindrical**

$$\int_0^{2\pi}\int_1^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2\cdot r\,dz\,dr\,d\theta$$

spherical



$$\int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_{\frac{1}{\sin\phi}}^{2}\rho^2\sin^2\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$